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Sylver Coinage Game

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In this paper, we will discuss the rules and strategies of the Sylver Coinage game along with implications of discovered theorems and corollaries. In particular, discussion about what moves provide a player with a winning strategy. Examples of gameplay will be provided along with proofs for all theorems and formulas. We will conclude with discussion for future research into unknown values of gameplay.

James Joseph Sylvester posed the following question: A man possesses a large quantity of stamps of only two denominations: 5-cent stamps and 17-cent stamps. What is the largest amount of postage which the man cannot make up with a combination of these stamps? Without much effort, we can simply list integers until we find five consecutive that he can obtain meaning that he can obtain all values higher than those five. In this case, he can make up 64, 65, 66, 67 and 68 but not 63. Therefore 63 is the largest amount that he cannot make up using 5-cent and 17-cent stamps. Note that he can obtain any number greater than 68 by adding the necessary amount of 5-cent stamps to one of the values between 64 and 68. Later we will discuss how to derive a general formula for this situation to help in situations where the numbers are too large to list.

First let's look at a game with a similar concept created by John Horton Conway that he named Sylver Coinage in honor of J.J. Sylvester. Two players take turns naming positive integers that are not attainable from the previous numbers chosen. The player that is forced to say 1 loses. Let's examine a basic game: Player 1 names #4. Player 2 cannot say any multiple of 4 so they choose #5. Using similar logic as our stamp problem above, we find the largest unattainable number is #11. In fact we can represent all the unattainable values visually using the following simple $4 \ x \ 5$ chart:

0	4	8	12	16
1	5	9	13	17
2	6	10	14	18
3	7	11	15	19

By using 4 rows, once a number is found to be attainable in a row every number following in that row is also attainable by adding some multiple of 4. This way we can see that

1, 2, 3, 6, 7, and 11 are the only unattainable numbers given the current position in our game of {4,5}. Player 1 has to choose one of these six numbers. They obviously don't want to choose #1 and if they choose #2, then Player 2 will choose #3 which will leave only #1 left for Player 1. The same logic is true if Player 1 were to pick #3, then Player 2 would choose #2 forcing Player 1 to pick #1. Therefore we can look at (2, 3) as a clique. Once one of them is chosen, the other player simply choses the other one for the win. This particular clique will hold for every possible game of Sylver Coinage.

This leaves Player 1 with 6, 7, or 11. Since $6 + 5 = 11$ and $7 + 4 = 11$, then if either is chosen it eliminates #11 as an option which makes (6, 7) a clique. If Player 1 chooses either of them, then Player 2 will choose the other in order to leave Player 1 with choices of 1, 2, or 3 which we have already discussed as losing choices for Player 1. Therefore Player 1 must choose #11 which forces Player 2 to choose a number in a clique which will give Player 1 the winning strategy. Given this information, it is clear that Player 2 should not have chosen #5 on their first turn. So let's analyze a different game where Player 2 chose #6 instead of #5. We have the following chart at this point in the game of {4,6}:

0	**4**	**8**	**12**	**16**	**20**
1	**5**	**9**	**13**	**17**	**21**
2	**6**	**10**	**14**	**18**	**22**
3	**7**	**11**	**15**	**19**	**23**

This new game gives Player 2 the winning strategy as every number belongs to a clique meaning that Player 2 has a response for any number Player 1 chooses. (5, 7), (9, 11), (13, 15), (17, 19),... are all cliques at this point in the game. If Player 1 chooses a value H, then Player 2 has to look at H-1. If H-1 is a multiple of 4, then they pick H+2. If H-1 isn't a multiple of 4, then they pick H-2. As long as Player 2 continues this strategy for each value that Player 1 picks not including #1, #2, or #3, then Player 2 will eventually win the game. Since we have found a winning strategy for Player 2 when Player 1 starts the game with #4, then we see that #4 is a bad starting number for Player 1. This discovery helps us to start deciding some of the answers to the following questions:

Which starting values give Player 1 a winning strategy?

Which starting values give Player 2 a winning strategy?

Theorem – R.L. Hutching

If A and B are coprime (g = 1) and $\{A, B\} \neq \{2,3\}$, then {A,B} gives Player 1 a winning strategy.

In fact, R.L. Hutchings proved a theorem that states: Given a Sylver Coinage game at position {A,B}, Player 1 has a winning strategy if A and B are co-prime and $\{A, B\} \neq \{2,3\}$. Looking at our example {4,5}, we found that Player 1 had the winning strategy and we can see that 4 and 5 are co-prime. Hutchings proves this by using strategy stealing. Hutchings looks at the largest unattainable number or the Frobenius number, a formula that we will prove later, which is $F = A * B - A - B$. If this number gives a winning strategy to Player 1 then the theorem is true. If this number is not a winning strategy for Player 1, then that means there is a value S, obviously less than F, that is a winning strategy for Player 2 given the game at the point {A,B,F}. In this scenario, Player 1 should steal Player 2's strategy and choose S instead of F giving Player 1 the winning strategy. This strategy is accomplished because F is attainable given {A,B,S} which we will also prove later. The problem with this theorem is while it does prove that there is a winning strategy for Player 1 given co-prime A and B; it does not give an algorithm for how to achieve that winning strategy. But before we delve into this conundrum, let's prove our assertions.

Sylvester's Formula (1882)

If A and B are relatively prime positive integers, then the largest number that is not a sum of nonnegative multiples of A and B is:
$(A-1)(B-1)-1 = A * B - A - B$.

First off, we need a general formula for a situation similar to the one presented at the beginning of this paper dealing with stamps. Let A and B be two co-prime integers greater than 1. What is the largest number that could not be expressed as a linear combination $mA + nB$, with non-negative integers m and n? In order to find/prove this formula, we need to use the help of The Euclidean Algorithm and one of its' corollaries: Suppose A and B are relatively prime positive integers, then given an integer k, there exist integers m and n such that $k = mA + nB$. From this we can see that

$$k = (m + B)A + (n - A)B \text{ or } k = (m - B)A + (n + A)B$$

are also representations for k. In fact, all representations of k are given by:

$$k = (m + jB)A + (n - jA)B \text{ with integer j,}$$

which we can choose so that our coefficient of A falls within the interval $[0, B - 1]$.

Now that k can be uniquely written as $k = mA + nB$ where $0 \leq m \leq B - 1$, let's show that k is representable only if $n \geq 0$. Suppose k is representable, then $k = jA + lB$ for some nonnegative integers j and l. If $0 \leq j \leq B - 1$ then we are done; otherwise, we subtract enough multiples of B from j such that $0 \leq m = j - qB \leq B - 1$. Then the coefficient l has to be adjusted to $n = l + qA$, which is positive. Since n has to be positive for a representable k when $0 \leq m \leq B - 1$, then this also implies that every integer $k \geq AB$ is representable.

Now we have our foundation to prove that the largest number that cannot be expressed by co-prime A & B is $F = A * B - A - B$. We have denoted this number F as it is called the Frobenius number after mathematician Ferdinand Frobenius and his work on the coin problem. Since k is representable when $n \geq 0$ and $0 \leq m \leq B - 1$, then we are trying to maximize m and n while keeping k not representable. Therefore we make $m = B - 1$ and $n = -1$, giving us the formula:

$$k = (B - 1)A + (-1)B = A * B - A - B$$

<u>Theorem – J.J. Sylvester (1882)</u>

If A and B are relatively prime positive integers, then the number of non-representable integers less than A*B is $\frac{1}{2}(A - 1)(B - 1)$.

Now that we have a general formula for the largest unattainable (Frobenius) number given co-prime A & B, let's find a formula for how many unattainable numbers exist in this scenario. First we show that if k is not divisible by A or B and is representable, then $AB - k$ is not representable. Note that $0 < k < AB$. So we suppose that k is representable, which means $k = mA + nB$ for some nonnegative integers m and n with $0 < m \leq B - 1$. This means that $AB - k = AB - mA - nB = (B - m)A - nB$, which shows us that $AB - k$ can be written as $AB - k = gA + hB$ where $0 \leq g \leq B - 1$ and $h < 0$. Since $h < 0$, then $AB - k$ is not representable. With this fact, we know that exactly half of the values between 1 and $AB - 1$ are representable, because if k is representable then $AB - k$ is not and vice versa. Therefore since there are $AB - A - B + 1 = (A - 1)(B - 1)$ integers between 1 and $AB - 1$ that are not divisible by A or B, then the number of non-representable integers given co-prime A and B is $\frac{1}{2}(A - 1)(B - 1)$.

The last concept we need to prove is that the Frobenius number F is attainable by co-prime values of A, B, and S where $0 < S < AB$. We just showed that if S is not representable by A and B, then $AB - S$ is representable which means it can be written as $AB - S = mA + nB$ for nonnegative integers m and n. Rewriting $AB = mA + nB + S$, which means we can write the Frobenius number as the following:

$$F = AB - A - B = mA + nB + S - A - B = (m-1)A + (n-1)B + S$$

which shows that F is attainable by A, B, and S, since m and n are nonnegative integers.

Hutching's Theorem – Corollaries

1) If $A \geq 5$ is a prime number, then {A} gives Player 1 a winning strategy.
2) If A is a composite number not of the form 2^x3^y, then {A} gives Player 2 a winning strategy.

With this proof we have shown why Hutching's explanation of his proof using a strategy stealing concept does in fact work, however he does not explain how to find the winning strategy. The truth is that no one actually knows how to find this strategy, we just know that there is one. Hutchings did however help us to find some specific facts about Player 1's starting moves. For instance, using his theorem that Player 1 has a winning strategy if A and B are co-prime and $\{A, B\} \neq \{2,3\}$; we can discover the corollary that if Player 1 starts the game with a prime number greater than 3, then Player 1 has a winning strategy because Player 2 has to select a number B that would be co-prime to A. Of course we would again run into the problem of how to implement that winning strategy for non-trivial games. Using this corollary, we can produce a second corollary that states that Player 2 would have the winning strategy if Player 1 starts the game with a composite number not of the form 2^x3^y, where $x, y \geq 0$, because Player 2 can now select a prime factor of Player 1's choice forcing Player 1 to have to pick a co-prime number to the first two choices. Using these two corollaries and some gameplay logic, we have information on the following openings up to 50:

Player 1 has a winning strategy if the first value played is:

5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, ...

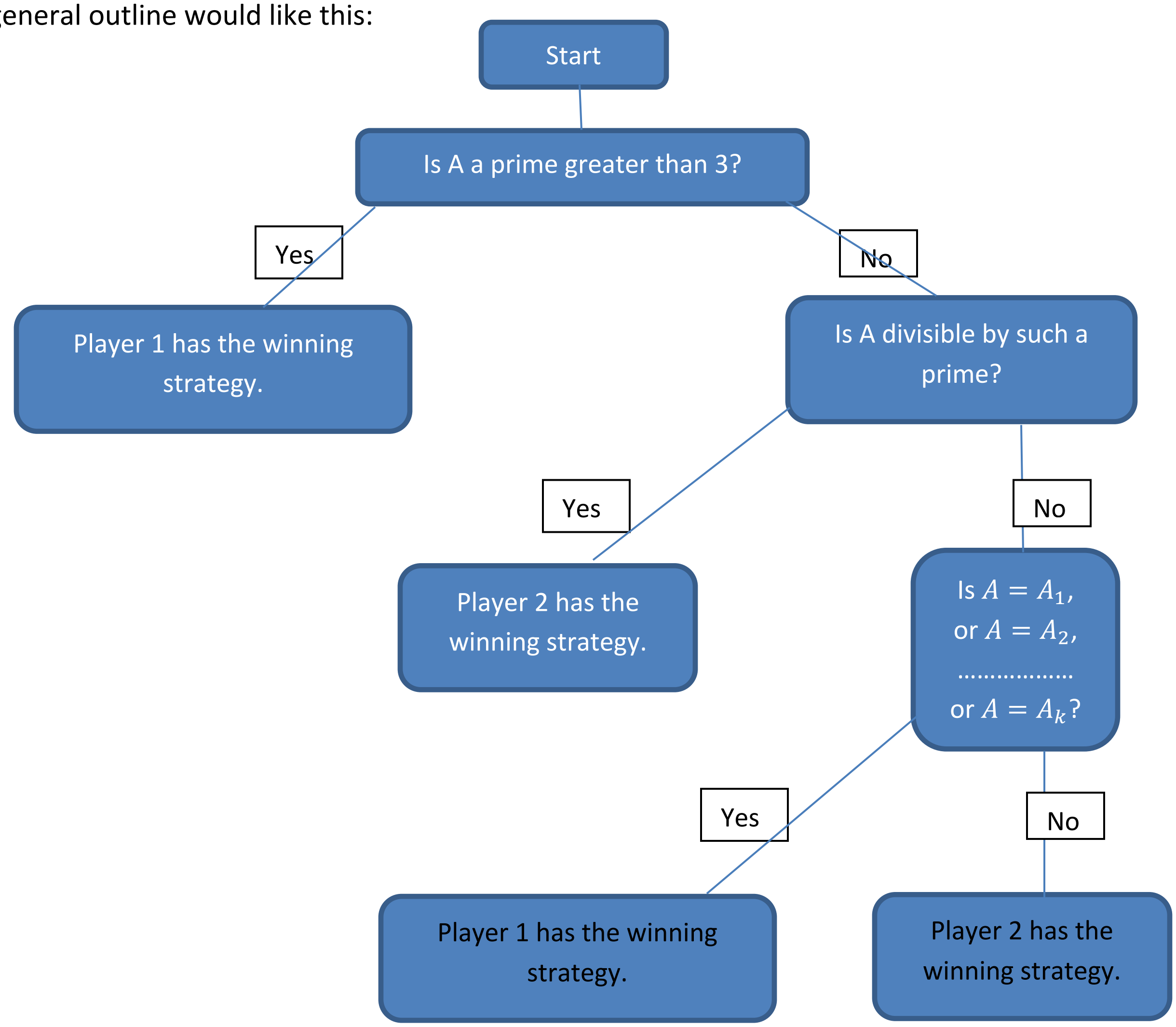
Player 2 has a winning strategy if the first value played is:

1, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 20, 21, 22, 25, 26, 28, 30, 33, 34, 35, 38, 39, 40, 42, 44, 45, 46, 49, 50, …

Values that are unknown as far as who has the winning strategy:

16, 18, 24, 27, 32, 36, 48, …

It should be noted that all unknown values are of the form 2^x3^y. It should also be noted again that although we know that there is a winning strategy for a player; we don't necessarily know what that strategy says to do unless it is a relatively trivial game. This would be one of the goals for future research: create an algorithm to find the winning value that needs to be chosen in order to successfully win the game when it is known that a player has a winning strategy. The second goal would be to find who has the winning strategy when the game begins with one of the unknown values. This has actually lead to another frustrating discovery with Sylver Coinage. It has been argued that it can be proven that there is a way of programming a computer to find the outcome of any starting value, but it is not known how to actually create such a code. Since we have Hutching's corollaries that tell us what happens for most numbers, the main focus for this program is to identify numbers of the form 2^x3^y. The general outline would like this:



Recall also that even if we could produce a viable program for a computer to calculate who has the winning strategy, we would still have the problem of how to go about implementing such a strategy. Like Hutching's theorem, we may know who has the winning strategy with optimal play, but no one has found how to determine what the optimal play is for any given game. Even with a specific non-trivial game, it can be nearly impossible to decide what is the optimal play. Your best play is to use these facts to prove to your opponent that there is a winning strategy for you and hope that they concede, but don't let on to the fact that you don't actually know how to implement it. Good Luck!

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